Type IIB Holographic Superfluid Flows

Daniel Areán^{a,b}, Matteo Bertolini^{a,b}, Chethan Krishnan^{b}
and Tomáš Procházka^{b}

^{a} International Centre for Theoretical Physics (ICTP)
Strada Costiera 11; I 34014 Trieste, Italy

^{b} SISSA and INFN - Sezione di Trieste
Via Bonomea 265; I 34136 Trieste, Italy

`arean,bertmat,krishnan,procht @sissa.it`

Abstract

We construct fully backreacted holographic superfluid flow solutions in a five-dimensional theory that arises as a consistent truncation of low energy type IIB string theory. We construct a black hole with scalar and vector hair in this theory, and study the phase diagram. As expected, the superfluid phase ceases to exist for high enough superfluid velocity, but we show that the phase transition between normal and superfluid phases is always second order. We also analyze the zero temperature limit of these solutions. Interestingly, we find evidence that the emergent IR conformal symmetry of the zero-temperature domain wall is broken at high enough velocity.

1   Introduction

In the last few years there has been an intense effort to model superconductor/superfluid phase transitions using the AdS/CFT correspondence. The basic observation that makes this industry possible is the fact that at finite charge density and at sufficiently low temperatures, an AdS black hole in the presence of a charged scalar field is unstable to the formation of hair [1]. Using the basic AdS/CFT dictionary [2, 3, 4], this gets easily interpreted as a superfluid-like phase transition in the dual field theory, cf. Weinberg [5].

Much of the work on holographic superconductors is done in the context of phenomenological models, along the lines of the proposal originally presented in [6]. This is based on the minimal set-up of a charged massive scalar minimally coupled to Einstein-Maxwell theory. While many interesting results can be obtained within this minimal framework (see [7, 8, 9] for reviews and references), such a bottom-up approach has some intrinsic limitations. Since the hope is that holographic constructions may eventually shed some light on some basic properties of high-T_c superconductors, it would be desirable to have a microscopic understanding of the underlying theory. This is something that phenomenological models, by definition, cannot offer. Secondly, they do not guarantee the existence of a quantum critical point in the phase diagram, which is instead expected to control the physics of high-T_c superconductors. Indeed, the phenomenological models that one typically works with have no potentials but the mass term. However, it is expected that to have an emergent conformal symmetry in the infrared in the zero temperature limit, one should have potentials that allow symmetry-breaking minima [10]. Recently, some progress has been made in this respect and several microscopic embeddings of holographic superconductors have been proposed in the framework of type IIB string theory [11], M-theory [12], and D7-brane models [13]. In these models, the potentials quite generically allow symmetry breaking vacua.

Most studies have also been performed in the probe approximation, which is a large-charge limit in which the backreaction of the matter fields on the gravitational field is negligible. While many interesting results can be obtained with such a simplified setup when the temperatures are near the phase transition, the analysis becomes less and less reliable at very low temperatures, where the backreaction is non-negligible. This prevents exploration of interesting low temperature phenomena: in particular, understanding the ground state of holographic superconductors is outside the regime of applicability of the probe limit. Therefore, it is useful to realize holographic constructions where the backreaction is taken into account. Progress in this direction began with [14], where a (numerical) backreacted solution for the phenomenological model of [6] was presented.

In trying to explore the phase diagram of holographic superconductors, an interesting direction was pursued in [15, 16] where the original holographic superfluid was studied in

the presence of a non-vanishing superfluid velocity (aka superfluid flow). Holographically this needs a non-trivial profile for a spatial component of the gauge field, besides the ever-present temporal component. The latter corresponds to a charge density and is necessary to have a phase transition in the first place (see [17] for a recent alternative proposal). Two interesting results obtained in [15, 16] were to show the existence of a critical velocity above which the superfluid phase ceases to exist, as expected for physical superfluids, and the existence of a tricritical point in the velocity vs. temperature diagram where the order of the phase transition changes from second to first. Moreover, it was noticed in [18, 19] that these solutions can be efficiently compared to 2+1-dimensional superconducting thin films or wires. They behave very much like superfluids, in that an applied external magnetic field does not get expelled as if the gauge field were not dynamical. The four-dimensional gravitational model of [15, 16] was further analyzed from this latter viewpoint in [19], where the system was in fact studied at fixed current rather than at fixed velocity. This choice allowed new checks, and remarkable agreement with some peculiar properties of real-life superconducting films (see [20]) was found.

All solutions presented in [15, 16, 19] have been obtained in the probe approximation. Hence, while being able to confront phenomena near or right below the critical temperature, not much could be said about the low temperature regime of such superfluid flows. This problem was addressed more recently in [21], where the backreaction of the phenomenological four-dimensional model of [15, 16] was obtained.

1.1 Summary of Results

In this paper we take some concrete steps forward in the above program on superfluid flows: we focus our attention on models with known microscopic embedding and symmetry breaking vacua, and work at the backreacted level. Specifically, we will describe a holographic super-fluid flow in four dimensions by means of a fully backreacted solution of a five-dimensional gravitational system whose action arises as a consistent truncation of type IIB string theory [11]. The effective theory is essentially Einstein-Maxwell theory with a Chern-Simons term, interacting with a complex charged scalar with a non-trivial potential. It can be obtained upon compactification of type IIB theory on an $AdS_5 \times Y$ geometry, Y being a Sasaki-Einstein manifold. Using the numerical solutions that we find, we analyze several aspects of the rich phase diagram of this system. In particular, we present the plots of the scalar condensate against temperature and its dependence on the superfluid velocity, analyze the nature of the phase transition computing the free energy difference between the superconducting and the normal phase, and give some predictions on the zero temperature limit.

As one would expect on physical grounds, we observe that for high enough velocity the

2

system stops superconducting. Interestingly, we find that for all velocities we have investigated the phase transition in these type IIB constructions is always second order. Hence, we do not find the tricritical point which characterizes the phase diagram of models with large charges. The same behavior was observed in the phenomenological but backreacted AdS_4 model of [21] for low values of the scalar charge (in fact, only for the case $q = 1$ in their notation). The persistence of the second order phase transition has been observed also in the unbackreacted case for large masses of the scalar in five dimensions [22]. We will have some more comments on this in section 4.

One of the advantages of having a fully backreacted model is that one can also investigate the low temperature limit. In the zero velocity case, it is known [23, 10] that the type IIB hairy black hole solution tends to a domain wall with an emergent conformal symmetry in the deep IR. (This is in contrast with the phenomenological model of [24] where the potential has only a mass term and no symmetry breaking minima, and the zero temperature limit generically does not lead to an IR AdS geometry.) When the velocity is turned on and it is high enough, we find evidence that the solution stops being AdS in the IR. This suggests that beyond some critical velocity the IR conformality is lost. Along the way, we also discuss the importance of the frame comoving with the superfluid flow in these results.

The rest of the paper is organized as follows. In section 2 we present the truncated type IIB five-dimensional action and the equations of motion. Our ansatz for the relevant fields, and the procedure we pursue to obtain numerical solutions with the desired features are discussed in section 3. Using these solutions, in section 4 we study the phase diagram of the superfluid flow. In particular, we analyze the nature of the phase transition as a function of the superfluid velocity. In doing so, we compute the free energy of the superfluid phase and compare it to that of the normal phase (which is described, holographically, by a Reissner-Nordstrom black hole with no scalar hair). Finally, in section 5 we study the $T \to 0$ limit of some geometrical quantities like the Ricci scalar and the Riemann tensor squared. We also study the variation of the superfluid fraction as the temperature is lowered. These analyses allow us to explore the nature of the ground state of holographic type IIB superfluid flows. The appendices contain more technical material which might help the reader in following our analytical and numerical computations more closely.

2   The IIB Set Up

In [11] a consistent truncation of type IIB supergravity was presented, which has the structure of an Einstein-Maxwell (plus Chern-Simons) system in five dimensions coupled to

a charged scalar field with a non-trivial potential. The action reads

$$
S_{IIB} = \int d^5x \sqrt{-g} \left[R - \frac{L^2}{3} F_{ab} F^{ab} + \frac{1}{4} \left(\frac{2L}{3} \right)^3 \epsilon^{abcde} F_{ab} F_{cd} A_e + \right.
$$

$$
\left. -\frac{1}{2} \left((\partial_a \psi)^2 + \sinh^2 \psi (\partial_a \theta - 2A_a)^2 - \frac{6}{L^2} \cosh^2 \left(\frac{\psi}{2} \right) (5 - \cosh \psi) \right) \right] . \quad (2.1)
$$

Here, $\epsilon^{01234} = 1/\sqrt{-g}$, and we have written the charged (complex) scalar by splitting the phase and the modulus in the form $\psi e^{i\theta}$. For later convenience we recall that the Abelian gauge field A is dual to an R-symmetry in the boundary field theory [11] and the scalar field has R-charge $R = 2$.

The matter equations of motion are

$$
\frac{1}{\sqrt{-g}} \partial_a \left(\frac{4}{3} L^2 \sqrt{-g} F^{ab} - \frac{8}{27} L^3 \sqrt{-g} \epsilon^{abcde} F_{cd} A_e \right) +
$$

$$
+ \frac{2}{27} L^3 \epsilon^{pqrsb} F_{pq} F_{rs} + 2 \sinh^2 \psi (\partial^b \theta - 2A^b) = 0 , \quad (2.2)
$$

$$
\frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} \partial^a \psi) - \frac{1}{2} \sinh 2\psi (\partial_b \theta - 2A_b)^2 +
$$

$$
+ \frac{3}{2L^2} \left(\sinh \psi (5 - \cosh \psi) - 2 \cosh^2 \left(\frac{\psi}{2} \right) \sinh \psi \right) = 0 . \quad (2.3)
$$

The Einstein equations can be written as

$$
R_{ab} - \frac{1}{2} g_{ab} R - \frac{2}{3} L^2 \left(F_{ac} F_b{}^c - \frac{g_{ab}}{4} F^{cd} F_{cd} \right) +
$$

$$
- \frac{1}{2} \Xi_{ab} + \frac{1}{4} g_{ab} \Xi_a{}^a - \frac{3}{2L^2} g_{ab} \cosh^2 \left(\frac{\psi}{2} \right) (5 - \cosh \psi) = 0 , \quad (2.4)
$$

with $\Xi_{ab} \equiv \partial_a \psi \partial_b \psi + \sinh^2 \psi (\partial_a \theta - 2A_a)(\partial_b \theta - 2A_b)$.

It is convenient to use the gauge invariance to shift away the angle θ and also write the various expressions in terms of covariant derivatives. This basically means that we set θ to zero in the above equations and use

$$
\nabla_a \left(\frac{4}{3} L^2 F^a{}_b - \frac{8}{27} L^3 \epsilon^a{}_b{}^{cde} F_{cd} A_e \right) + \frac{2}{27} L^3 \epsilon^{pqrs}{}_b F_{pq} F_{rs} - 4 \sinh^2 \psi \, A_b = 0 , \quad (2.5)
$$

$$
\nabla_a \nabla^a \psi - 2 \sinh 2\psi (A_b A^b) + \frac{3}{2L^2} \left(\sinh \psi (5 - \cosh \psi) - 2 \cosh^2 \left(\frac{\psi}{2} \right) \sinh \psi \right) = 0 , \quad (2.6)
$$

as the matter equations of motion. The leading terms in the scalar potential take the form

$$
V(\psi) = -\frac{12}{L^2} - \frac{3\psi^2}{2L^2} + \dots \quad (2.7)
$$

4

which have the immediate interpretation as the AdS cosmological constant and the scalar mass term. Typically, in a minimal phenomenological model the scalar potential has just the above two terms. Higher order terms affect mostly the very low temperature regime where the condensate becomes larger and thus the type IIB model can become substantially different from the minimal one. There are then two reasons as to why one should try and work out a fully backreacted solution for this type IIB model. The first is that the scalar has charge $R = 2$ and hence the probe approximation, which is a large charge scaling limit, is potentially inappropriate already at temperatures near the critical temperature. The second is that a backreacted solution would let one study the system in a regime (i.e. very low temperatures) where, as just noticed, the differences of the action (2.1) with respect to that of a phenomenological model, are more apparent.

Note that the scalar mass is $m^2 = -3$. In $d = 4$, this mass is in the range where the leading fall-off at the boundary, which is $\mathcal{O}(1/r)$, corresponds to a non normalizable mode. So, using the AdS/CFT map, we will interpret it as the source of the dual field theory operator $\mathcal{O}$. The subleading fall-off is $\mathcal{O}(1/r^3)$ and corresponds to a condensate for $\mathcal{O}$ (whose dimension will therefore be $\Delta = 3$). It is evident from the value of the R-charge and this fall-off that $\Delta = 3|R|/2$ and $\mathcal{O}$ is therefore a chiral primary [11].

3 Hairy Black Hole Solution

We want to construct a fully backreacted hairy black hole solution, holographically describing a superfluid flow. To achieve this we must keep the metric (also) unfixed and find a self-consistent solution for the metric, the gauge field and the scalar. To have a charged scalar condense, we need to turn on both the scalar and the time component of the gauge field in the bulk [1]. Moreover, to obtain a non-vanishing superfluid flow, we should break the isotropy in the boundary directions that was present in the original holographic superconductor construction of [11]. Indeed, the superfluid velocity in (say) the x-direction is captured by the leading fall-off of the bulk gauge field component A_x at the boundary, which should therefore have a non-trivial bulk profile. Altogether, this means that we need ψ, A_t and A_x to be non-trivial. Since we would like to work with ordinary as opposed to partial differential equations, we look for an ansatz where these are functions purely of the holographic direction r : fortunately, this turns out to be enough to obtain a solution. Consistency of the Einstein equations then demands that we choose a metric ansatz of the form

$$ds^2 = -\frac{r^2 f(r)}{L^2}dt^2 + \frac{L^2 h(r)^2}{r^2 f(r)}dr^2 - 2C(r)\frac{r^2}{L^2}dtdx + \frac{r^2}{L^2}B(r)dx^2 + \frac{r^2}{L^2}dy^2 + \frac{r^2}{L^2}dz^2 \,. \quad (3.1)$$

The metric contains four independent functions, $f(r), h(r), C(r)$ and $B(r)$. Together with the ansatz for the gauge field and the scalar

$$A = A_t(r) \, dt + A_x(r) \, dx \, , \qquad \psi = \psi(r) \, , \tag{3.2}$$

this will give rise to a set of seven independent equations for seven unknowns. Our ansatz here is essentially of the same form as the one in [21], albeit in one more dimension. This can be demonstrated by going over to an Eddington-Finkelstein form and working in a frame where the normal fluid considered in [21] is at rest.

Let us first notice that with this choice of ansatz the terms in the equations of motion (2.5) arising from the ϵ^{abcde} piece all vanish. A second important fact is that there are several scaling symmetries one should be aware of. In particular, the ambiguity in the units at the boundary for the time t and the distance along x translate to the following two scaling symmetries of the resulting equations

$$t \to t/\mathrm{a} \, , \quad f \to \mathrm{a}^2 f \, , \quad h \to \mathrm{a} \, h \, , \quad C \to \mathrm{a} \, C \, , \quad A_t \to \mathrm{a} \, A_t \, , \tag{3.3}$$

$$x \to x/\mathrm{b} \, , \quad B \to \mathrm{b}^2 B \, , \quad C \to \mathrm{b} \, C \, , \quad A_x \to \mathrm{b} \, A_x \, . \tag{3.4}$$

These are symmetries of the action and therefore of the equations of motion. Two further scaling symmetries of the system that we will use are

$$(r, t, x, y, z, L) \to \alpha(r, t, x, y, z, L) \, , \quad (A_t, A_x) \to (A_t, A_x)/\alpha \, , \tag{3.5}$$

$$r \to \beta r \, , \quad (t, x, y, z) \to (t, x, y, z)/\beta \, , \quad (A_t, A_x) \to \beta(A_t, A_x) \, . \tag{3.6}$$

The first scaling changes the metric by a factor α^2 and leaves the gauge field invariant, but its effect is to scale the action (2.1) by an overall constant factor α^2, therefore leaving the equations of motion unaffected. The second scaling is the usual holographic renormalization group operation in AdS, and it is easily seen that the metric, gauge field and the equations of motion are left invariant. Using the symmetries (3.5) and (3.6) we can scale the horizon radius r_H and the AdS scale L to unity. We will assume this has been done in what follows, unless stated otherwise.

The strategy we pursue to construct (numerically) our solution is as follows. First, using our ansatz, one can massage the equations of motion and end up with first order differential equations for f and h and second order differential equations for B, C, A_t, A_x and ψ. All in all we have then two first order and five second order equations resulting in twelve degrees of freedom. Therefore, to fix a solution we need twelve pieces of data.

We start by considering the fields (3.1)-(3.2) near the horizon $(r = r_H)$ and expand their several components Φ in a Taylor series as

$$\Phi = \Phi_0^H + \Phi_1^H (r - r_H) + \dots . \tag{3.7}$$

Requiring regularity of the solution at the horizon amounts to setting some specific coefficients to zero. To linear order in $(r - r_H)$, the expansion at the horizon takes the form

$$f = f_1^H(r - r_H) + ... \tag{3.8}$$

$$h = h_0^H + h_1^H(r - r_H) + ... \tag{3.9}$$

$$B = B_0^H + B_1^H(r - r_H) + ... \tag{3.10}$$

$$C = C_1^H(r - r_H) + ... \tag{3.11}$$

$$A_t = A_{t,1}^H(r - r_H) + ... \tag{3.12}$$

$$A_x = A_{x,0}^H + A_{x,1}^H(r - r_H) + ... \tag{3.13}$$

$$\psi = \psi_0^H + \psi_1^H(r - r_H) + ... \tag{3.14}$$

That is, demanding regularity is tantamount to setting f_0^H, C_0^H and ϕ_0^H to zero. Imposing now the equations of motion has the effect of putting further constraints on many coefficients, which all end up being determined by a small set of independent horizon data. It turns out that the coefficients can all be determined in terms of six independent data

$$(h_0^H, B_0^H, C_1^H, A_{t,1}^H, A_{x,0}^H, \psi_0^H). \tag{3.15}$$

This means that the solutions that we will find by integrating from the horizon will be a six-parameter family. All other coefficients are functions of these ones. One such relation which will be useful later is

$$f_1^H = (h_0^H)^2 \left(\frac{9}{4} + 2\cosh\psi_0^H - \frac{\cosh(2\psi_0^H)}{4} \right) - \frac{2(A_{t,1}^H)^2}{9}. \tag{3.16}$$

The next step is to integrate the solution from the horizon out to the boundary $(r \to \infty)$, starting with the free horizon data (3.15), trying a suitable ansatz for the asymptotics of the fields at the boundary. In fact, the asymptotic expansion in five dimensions is subtle because, as already noticed, the mass of the scalar is such that there is a non-normalizable mode. To accommodate a generic solution obtained by integration from the horizon, we therefore need to turn on the non-normalizable mode of the scalar as well at the boundary. The non-normalizable mode triggers further logarithmic terms in the asymptotic expansion, so we need to keep track of them as well. It turns out that a combined series expansion in both $1/r^n$ and $\log r/r^m$

$$\Phi = \sum_{n=0}^{\infty} \Phi_n \frac{1}{r^n} + \sum_{m=0}^{\infty} \Phi_m^l \frac{\log r}{r^m}, \tag{3.17}$$

works nicely.

Using a shooting technique we select, out of all possible solutions, those which match our physical requirements. In particular, we ask that the space be asymptotically AdS and that the source term for the field theory operator dual to the scalar field be vanishing, since we want the $U(1)$ breaking to be spontaneous.

We have found that the following asymptotic expansion solves the equations of motion[1], while being general enough to match the curves arising from the integration from the horizon

$$f = h_0^2 + \frac{f_4}{r^4} + \frac{f_4^l}{r^4} \log r + ... , \qquad h = h_0 + \frac{h_2}{r^2} + \frac{h_4}{r^4} + \frac{h_4^l}{r^4} \log r + ... , \quad (3.18)$$

$$B = B_0 + \frac{B_4}{r^4} + \frac{B_4^l}{r^4} \log r + ... , \qquad C = C_0 + \frac{C_4}{r^4} + \frac{C_4^l}{r^4} \log r + ... , \qquad (3.19)$$

$$A_t = A_{t,0} + \frac{A_{t,2}}{r^2} + \frac{A_{t,2}^l}{r^2} \log r + ... , \quad A_x = A_{x,0} + \frac{A_{x,2}}{r^2} + \frac{A_{x,2}^l}{r^2} \log r + ... , \quad (3.20)$$

$$\psi = \frac{\psi_1}{r} + \frac{\psi_3}{r^3} + \frac{\psi_3^l}{r^3} \log r + ... \qquad (3.21)$$

Of course, not all of the above coefficients are independent. We relegate the explicit expressions for the dependent ones to Appendix A. We merely note that when the non normalizable mode ψ_1 is set to zero, the expressions are such that all the logarithmic pieces vanish as expected. It can also be seen that the independent parameters at the boundary can be taken to be

$$(h_0, f_4, B_0, B_4, C_0, C_4, A_{t,0}, A_{t,2}, A_{x,0}, A_{x,2}, \psi_1, \psi_3) . \qquad (3.22)$$

To get asymptotically AdS solutions, we must set B_0, h_0 to 1 and C_0, ψ_1 to zero. The scaling symmetries can be used to accomplish the first two conditions, whereas we need to shoot for the last two. We are therefore left with eight independent boundary data. They are

$$(f_4, B_4, C_4, A_{t,0}, A_{t,2}, A_{x,0}, A_{x,2}, \psi_3) . \qquad (3.23)$$

We see here that the physical requirements we impose at the boundary do not fix as many integration constants of the ODE system, as the regularity conditions at the horizon does. This means that for the solutions that we obtain, there are hidden relations between the boundary data. Concretely, since there are only six independent pieces of horizon data this gives us relations between the above eight variables, which will then be used to study the phase diagram of the boundary theory.

An important quantity for studying the thermodynamics of the system is of course the superfluid temperature. This corresponds to the black hole Hawking temperature, T. From

---

[1]What we do is to plug this expansion into the EoMs and demand that the result be zero order-by-order. We find that either this is satisfied identically or that the resulting relations can be interpreted as the definitions of higher order terms in the expansion.

the structure of the metric (3.1) we easily get

$$T = \frac{r_H^2 \, f'(r_H)}{4\pi \, L^2 \, h(r_H)} \,, \tag{3.24}$$

which is then also determined in terms of our horizon data. After some simple algebra, recalling we have set $r_H = 1$ and using the horizon relation (3.16), we get

$$T = \frac{1}{4\pi} \left[h_0^H \left(\frac{9}{4} + 2\cosh\psi_0^H - \frac{\cosh(2\psi_0^H)}{4} \right) - \frac{2(A_{t,1}^H)^2}{9 h_0^H} \right] \,. \tag{3.25}$$

4 Superfluid Flow Phase Transition

We plot the result for the condensate versus the temperature in figure 1, for different values of the superfluid velocity. For the rest of the paper, we will introduce the notation

$$\mu \equiv A_{t,0}, \quad \langle \mathcal{O} \rangle \equiv \sqrt{2}\,\psi_3, \quad \xi \equiv \frac{A_{x,0}}{A_{t,0}} \,, \tag{4.1}$$

where μ is the field theory chemical potential, $\mathcal{O}$ the (condensing) chiral primary operator, and ξ the superfluid velocity in units of the chemical potential. When we work in an ensemble with fixed chemical potential, the meaningful (dimensionless) quantities relevant for the condensate plot are

$$\frac{T}{\mu} \quad \text{and} \quad \frac{\langle \mathcal{O} \rangle}{\mu^3} \,. \tag{4.2}$$

In constructing the plots, we have also rescaled by the (velocity-dependent) factor $\sqrt{1-\xi^2}$, which is nothing but the relativistic boost factor.

From the form of the curves in figure 1, it is evident that there is a phase transition to a hairy black hole at low temperatures. As expected, the critical temperature decreases as the velocity is increased. For instance, for $\xi = 1/2$ (which is the highest velocity we have investigated) we observe that $T_c(\xi = 1/2) = 0.067\,T_c(\xi = 0)$. It is clear from the condensate plot that the superfluid phase cannot exist for velocities that are much higher than this.

One can compare the free energy of the normal phase (which corresponds to a Reissner-Nordstrom black hole with no hair) and the hairy/superfluid phase to see that the superfluid phase is favored when it exists. We collect some details of the free energy computation in appendix D, while figure 2 contains the free energy comparison between the superfluid phase and the normal phase at the same value of T/μ. In terms of S_{ren} defined in eq. (B.11), the precise quantities we plot are

$$\frac{S_{\text{ren}}}{\mu^4 \, \text{Vol}_4} \equiv \frac{\Omega}{\mu^4} \quad \text{vs.} \quad \frac{T}{\mu} \,. \tag{4.3}$$

The plot demonstrates that the phase transition stays second order for all values of the velocity, up to our numerical precision. This should be contrasted to the unbackreacted
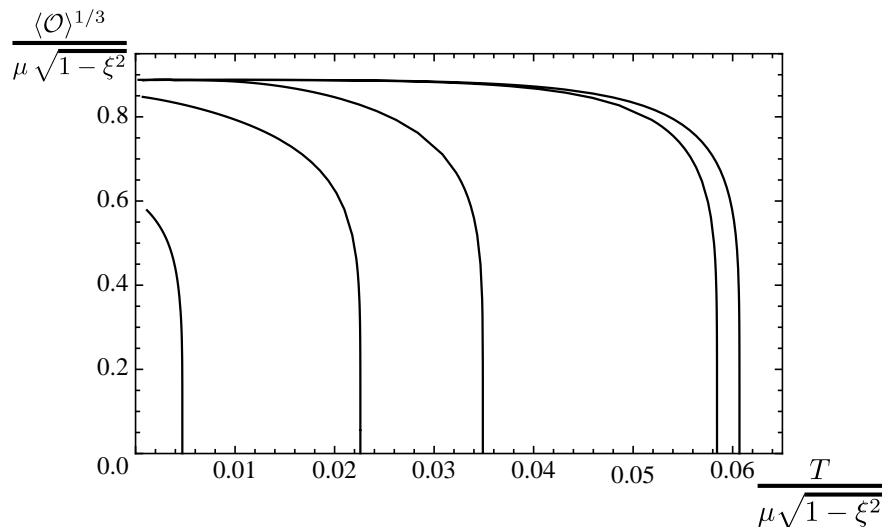
9

Figure 1: Condensate plots for various values of the velocity $\xi = 0, 0.1, 0.33, 0.4, 0.5$ (from right to left). The zero velocity case, $\xi = 0$, which we report for ease of comparison, precisely agrees with existing results in the literature [11].

cases previously considered in the literature, where the phase transition typically changes to first order for high enough values of the velocity [15, 16, 19]. In [21] a backreacted superfluid in AdS_4 was considered and it was found that for low enough values of the charge of the scalar field, the phase transition remained second order. Our type IIB system seems to be analogous to this latter scenario: the (R-)charge of the scalar in our case is fixed by the IIB construction to be 2 and it is plausible that this is a low enough value so that the transition remains second order all through.

In [22], the phases of the (unbackreacted) superfluid for various values of the masses of the scalar field in AdS_5 were investigated and it was found that for high enough mass, there is always a second order transition close to the normal phase. Since the probe limit is a large charge limit, we should expect a similar structure also in the backreacted case when the charge is large. That is, when the charge and the mass are both large, we should expect a persistent second order transition. In our IIB case, we are exploring the opposite limit, namely low (R-)charge and low mass (since the charge and mass are related for chiral primaries). Again, we find that the second order transition exists irrespective of the velocity. Based on these observations, it is tempting to make the suggestion that whenever the mass and charge are scaled together in some appropriate way, the second order transition persists for all velocities. Of course, to make and/or establish a precise statement along these lines will require a much more thorough exploration of the masses and charges of the scalars than we have undertaken here. Moreover, as already noticed, the persistence of the second order transition was also found in the AdS_4 case for small charges and small mass [21], while it
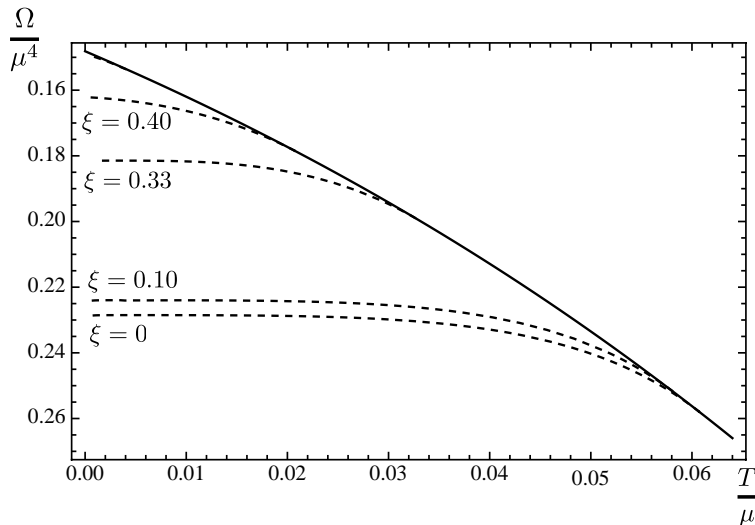
10

Figure 2: Free energy plots for various velocities $\xi = 0, 0.1, 0.33, 0.4, 0.5$ (dashed lines from bottom to top). The RN-AdS black hole is also presented for comparison (solid line). The plots show that for any velocities the phase transition is second order. The apparent overlap of the $\xi = 0.5$ curve with the normal phase, is an artifact of the resolution of the figure.

was found not to exist for any value of the mass in the probe limit [22]. So it is clear that the appropriate statement, if it exists, will have to be dimension-dependent.

5   Zero Temperature Limit

One of the advantages of having a fully backreacted solution is that one can reliably go to the zero temperature limit. At zero velocity, the zero temperature solution is expected to be described by a domain wall, corresponding to the symmetry-breaking vacuum of the scalar potential that restores conformal symmetry in the IR. Such domain wall solution was constructed in [23], and conjectured to correspond to the ground state of the type IIB holographic superconductor. Since we have here fully backreacted solutions at non-zero velocity, a natural question one would like to answer is whether and how such IR behavior gets modified when the superfluid flows.

As a warm-up, and for later comparison, let us first consider the static case. A preliminary check one can perform is to see whether for $\xi = 0$ our condensate value tends to the condensate value found in [23]. This is indeed the case: for the lowest temperature point ($T/\mu = 3.05 \cdot 10^{-4}$), our condensate in the normalizations of [23]

$$\langle \mathcal{O} \rangle_{DW} \equiv \frac{\psi_3}{(2\mu/\sqrt{3})^3} \tag{5.1}$$

is ≈ 0.3215, which is close enough to the zero-temperature value of ≈ 0.322 found in [23].
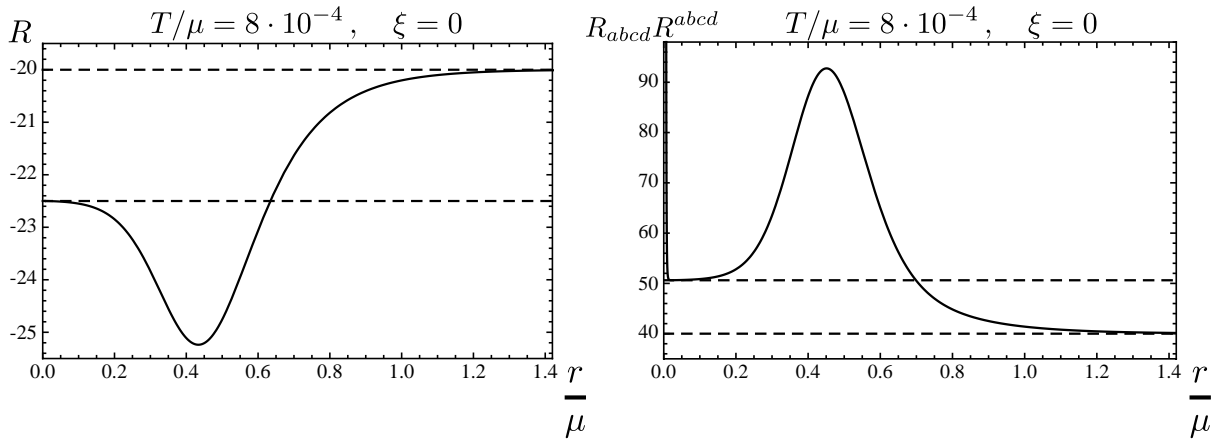
11

Figure 3: Ricci scalar R and $R_{abcd}R^{abcd}$ as a function of the radial coordinate near the horizon, at zero superfluid velocity. The horizontal dashed lines mark the corresponding values of R and $R_{abcd}R^{abcd}$ for the UV and IR AdS geometries.

Even without explicitly constructing the domain wall solution, one can find evidence for its existence by investigating the horizon values of the curvature scalars R and $R_{abcd}R^{abcd}$. This strategy was adopted in [12] for superconductors in M-theory, and it was found that these curvature scalars on the horizon go to the AdS_4 values expected from a domain wall solution with a symmetry-breaking minimum in the IR. We can do the same computation here, and we do find evidence that the solution has an emergent AdS_5 in the IR with the correct length scale. Note that the IR AdS scale, as determined by the symmetry-breaking vacuum [23] is $L' = \frac{2^{3/2}}{3}$ where we have set $L = 1$ in the UV. Using the fact that the Ricci scalar for AdS_5 is $-20/L^2$, we find that the predicted value is -22.5 in the IR. A similar computation using the $R_{abcd}R^{abcd} = 40/L^4$ shows that in the zero temperature limit we should get the value 50.625. We plot the results for both curvature scalars in figure 3. The plots clearly demonstrate that at low temperatures the curvatures indeed stabilize to the expected domain wall values in the infrared.

The behavior of $R_{abcd}R^{abcd}$ deserves a closer look, however. A distinctive feature of the present five-dimensional case, as compared to the four-dimensional model of [12], is that $R_{abcd}R^{abcd}$ stabilizes to the domain wall value close to the horizon, but it starts increasing as the radius is further reduced. At the horizon its value is (of course) finite, but is well on its way to the divergence at the singularity inside the horizon[2]. Note that in order to

---

[2]This sharp ascent in the curvature scalars close to the horizon is not a peculiarity of the broken phase: it is also there in the normal phase. For instance, $R_{abcd}R^{abcd}$, whose expression for the normal phase Reissner-Nordstorm black hole we report in eq. (D.4), has a similar sharp ascent at the horizon, while remaining finite there. On the other hand, the AdS_4 case is somewhat special in that the Ricci scalar is a constant in the normal phase due to the tracelessness of the electromagnetic stress tensor in four dimensions.

make the connection with the domain wall, what we really need is the emergence of an AdS_5 throat of the correct length scale at *zero* temperature, and our plots give evidence for that. Figure 4 reports the behavior of $R_{abcd}R^{abcd}$ zooming in near the horizon region for different temperatures. Happily, as the temperature is lowered the stabilized region of the plot gets closer to the horizon and asymptotes to the expected AdS_5 value of 50.625.
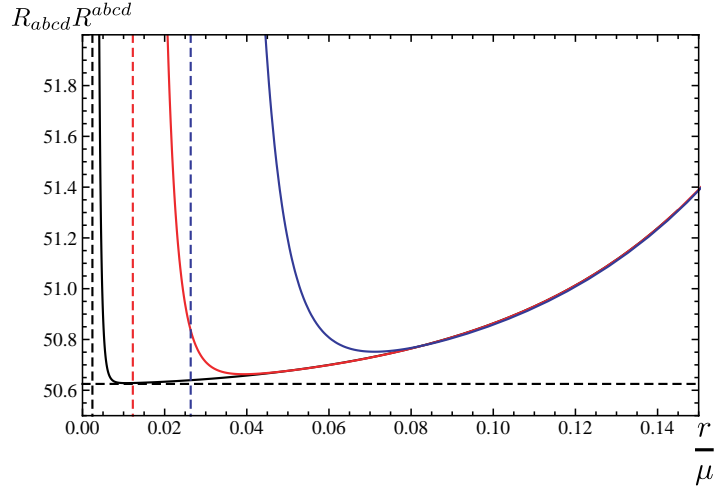

Figure 4: Behavior of $R_{abcd}R^{abcd}$ near the horizon for different temperatures, from left to right: $T/\mu = 3.05 \cdot 10^{-4}$ (black), $1.55 \cdot 10^{-3}$ (red), $3.33 \cdot 10^{-3}$ (blue). The dashed vertical lines correspond to the corresponding horizon radii. The stabilized (i.e. domain wall) value $R_{abcd}R^{abcd} = 50.625$ is indicated by the dashed horizontal line.

Let us now consider the cases with velocity, $\xi \neq 0$. We report in figure 5 the plot for the Ricci scalar vs. radius for different superfluid velocities (including the zero-velocity case, to ease the comparison) and in figure 6 that for $R_{abcd}R^{abcd}$. The presence of a new scale means that there is a possibility that the emergent conformal symmetry in the IR is broken. While for low velocities our plots suggest that the same IR fixed point as the static case is recovered, interestingly enough, we find that for high enough velocities the conformal symmetry of the solution is indeed broken and the curvature scalars diverge without any stabilization whatsoever. This is analogous to the phenomenological models with no quantum critical point in the IR. The conclusion seems to be that the solutions do not stabilize to the conformal quantum critical point when the velocity is high enough.

While we have not performed an exhaustive scan of velocities in this paper, it would be interesting to see for what precise value of the velocity this qualitative change happens, and study the precise nature of the phases and phase transitions, if any, there. From our analysis, it appears that the regime where this transition happens is between $\xi = 0.33$ and
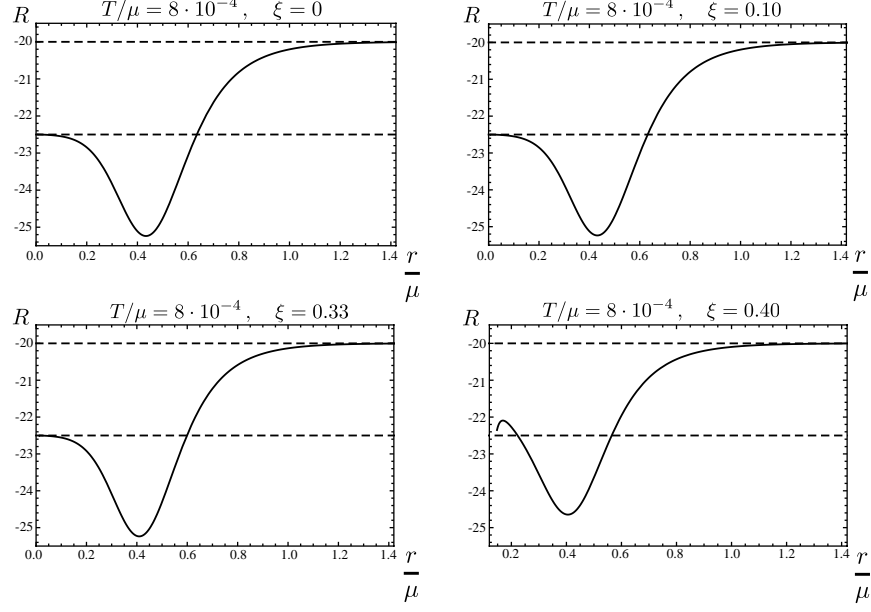
13

Figure 5: Ricci scalar R as a function of the radial coordinate near the horizon for a low temperature. The horizontal dashed lines indicate the corresponding values of R for the UV and IR AdS geometries.
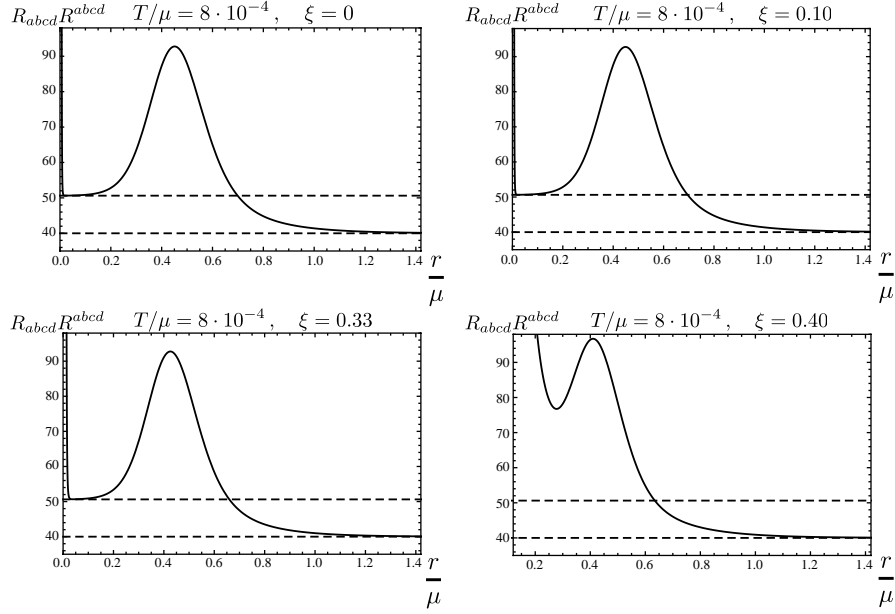

Figure 6: $R_{abcd}R^{abcd}$ as a function of the radial coordinate near the horizon for a low temperature. The horizontal dashed lines indicate the corresponding values of $R_{abcd}R^{abcd}$ for the UV and IR AdS geometries. The stabilization to the IR value, when it happens, holds till very close to the horizon.
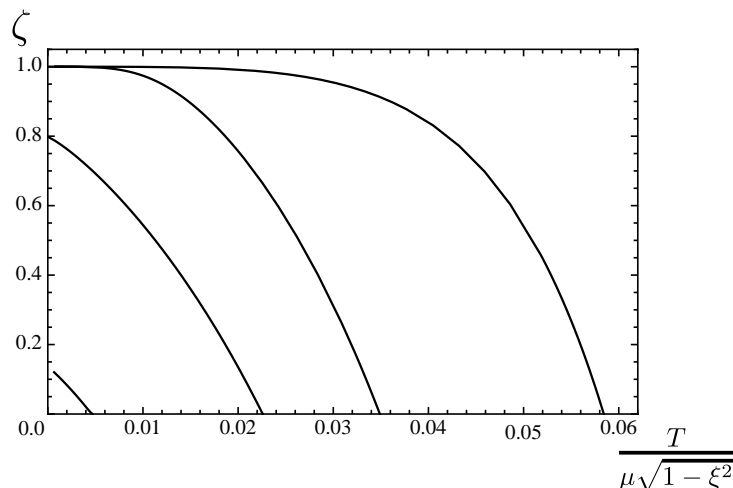
14

Figure 7: Plots of the superfluid fraction vs. temperature for various values of the velocity $\xi = 0.1,\ 0.33,\ 0.4,\ 0.5$ (from right to left).

$\xi = 0.4$. Related to this is the observation that the condensate

$$\frac{\langle \mathcal{O} \rangle^{1/3}}{\mu \sqrt{1 - \xi^2}} \tag{5.2}$$

that we plotted earlier, tends to the same value at the horizon for all values of the velocity, for small enough velocity. This is again indicative of a quantum phase transition: there is a change in the nature of the solution as we tune an order parameter at zero temperature. The results we find are consistent with the idea that the phase structure in the temperature-velocity plane is determined by the quantum critical point. It is intriguing that the relevant condensate seems to be measured in units of chemical potential as seen in a frame comoving with the superfluid flow. For a timelike vector, which for us is the superfluid velocity 4-vector, the time component in the rest (i.e., comoving) frame is nothing but its norm. Therefore, since we want to plot a scalar quantity for the dimensionless condensate, this is the natural choice. But unlike in the case of an ordinary fluid where the fluid velocity can be interpreted as arising from a boost of a static black hole, here the anisotropic part of the metric does not seem to have such a simple interpretation in the bulk. We intend to come back to some of these questions in the near future.

Another quantity of interest[3] in understanding the zero temperature limit is the superfluid fraction ζ. It corresponds to the ratio between the charge density of the superfluid flow and the total charge density of the system. In appendix C, following [21], we elaborate on the interpretation of the boundary theory in terms of a two-fluid model and compute the expression of the superfluid fraction in terms of the fall-offs of the bulk fields, eqs. (3.18)-

[3]We thank Julian Sonner for raising this point.

(3.21). The result is

$$\zeta = -\frac{A_{x,2}\,C_4}{A_{t,2}\,B_4}.$$

(5.3)

This quantity is interesting because from the curves in figures 1 and 2 of [21] we see that for the AdS_4 case its behavior near zero temperature captures some interesting aspects of the nature of the phase transitions. More specifically, together with our results in this paper (see figure 7), we are lead to conjecture that $\zeta \to 1$ at zero temperature for all velocities where the rescaled condensate value at zero temperature tends to its value at zero velocity. From the evidence presented in [21] one could think that the zero temperature limit of the superfluid fraction is correlated with the existence or not of a first order phase transition at high enough velocity. However, in our case we have an explicit situation where we see a consistently second order phase transition where the limiting value of the condensate at zero temperature changes qualitatively as we tune the velocity. Remarkably, we find that $\zeta \to 1$, only in those cases where the zero temperature condensate value $\frac{\langle \mathcal{O} \rangle^{1/3}}{\mu\sqrt{1-\xi^2}}$ takes its corresponding value at zero velocity. Since this condensate value captures the existence or not of the (anisotropic) domain wall, the natural conjecture is that $\zeta = 1$, for the domain wall when it exists. Notice that $\zeta \to 1$ is what one would expect for the ground state of a superfluid flow. What we have basically demonstrated then is that three quantities (namely the curvature scalar(s), the rescaled condensate and the superfluid fraction) undergo a qualitative change at the same velocity, as we tune the velocity. We believe this is strong evidence for the existence/non-existence of the domain wall as we go through that velocity.

Despite the evidence we have presented, it should be borne in mind that the preservation of the conformal symmetry for low velocities is not fully established. Unlike in the zero velocity domain wall examples discussed in the literature, we have not constructed an explicit solution that has emergent conformal symmetry in the IR in the cases with (low) velocity. However, the fact that the curvature scalars and the condensate (5.2) stabilize to their respective zero velocity values (within our numerical precision), is an indication that this might indeed be the case. One another caveat that we emphasize here is that the perturbative stability of these consistent truncations in the zero temperature limit is not settled. In particular, when the Sasaki-Einstein manifold is a sphere, instabilities are known to exist in the zero temperature domain wall solution [25, 26][4]. It is possible that for a more complicated choice of Sasaki-Einstein space (which is indeed what we need to have here anyway, in order to let the scalar chiral primary we focus on to be the operator responsible for the black hole phase transition [11]) the five-dimensional theory is stable. It is also interesting that the simple stringy consistent truncations do give rise to scalar potentials with symmetry breaking vacua, resulting in an emergent conformal symmetry in the IR at zero temperature.

---

[4]A related instability was recently shown to exist also in M-theory [27] for a similar consistent truncation for the ground state of a 2+1 dimensional superconducting system [23, 28].

This is precisely what one expects in the zero temperature limit of a high-T_c superconductor, which is believed to be governed by a quantum critical point. So our expectation is that in the (unlikely?) event that no Sasaki-Einstein truncation can be made stable, these models should still capture some generic features of a holographic superfluid with emergent conformal symmetry in the IR[5].

Acknowledgments

A   Asymptotic Relations

In this appendix, we present the relations defining the dependent coefficients in the asymptotic expansion in the IIB case

$$f_4 = \frac{1}{48C_0^2}\Big(96C_0C_4h_0^2 + 48B_4h_0^4 + 96C_0^2h_0h_4 + 96B_0h_0^3h_4 + C_0^2h_0^2\psi_1^4 + B_0h_0^4\psi_1^4 +$$
$$+24C_0^2h_0^2\psi_1\psi_3 + 24B_0h_0^4\psi_1\psi_3 - 12h_0^4L^4\psi_1^2A_{x,0}^2 + 24C_0h_0^2L^4\psi_1^2A_{x,0}A_{t,0} + 12B_0h_0^2L^4\psi_1^2A_{t,0}^2\Big),$$

(A.1)

$$f_4^l = \frac{1}{3C_0^2}\Big(-C_0^2h_0^2\psi_1^4 - B_0h_0^4\psi_1^4 + 3B_0h_0^2L^4\psi_1^2A_{t,0}^2 +$$
$$+C_0^2(h_0^2\psi_1^4 - 3L^4\psi_1^2A_{t,0}^2) + B_0h_0^2(h_0^2\psi_1^4 - 3L^4\psi_1^2A_{t,0}^2)\Big),$$

(A.2)

$$h_2 = -\frac{h_0\psi_1^2}{12}, \quad h_4^l = \frac{h_0^2\psi_1^4 - 3L^4\psi_1^2A_{t,0}^2}{6h_0}, \quad B_4^l = L^4\psi_1^2A_{x,0}^2, \quad C_4^l = -L^4\psi_1^2A_{x,0}\psi_0,$$

(A.3)

$$\psi_3^l = -\frac{2(C_0^2\psi_1^3 + B_0h_0^2\psi_1^3 + 3h_0^2L^4\psi_1A_{x,0}^2 - 6C_0L^4\psi_1A_{x,0}A_{t,0} - 3B_0L^4\psi_1A_{t,0}^2)}{3(C_0^2 + B_0h_0^2)},$$

(A.4)

$$A_{t,2}^l = -\frac{3\psi_1^2A_{t,0}}{2}, \quad A_{x,2}^l = -\frac{3\psi_1^2A_{x,0}}{2}.$$

(A.5)

[5]We thank Nikolay Bobev and Chris Herzog for a discussion on this point.

These are the general expressions when $\psi_1 \neq 0$. Our primary interest will be to shoot for the case $\psi_1 = 0$, in which case all of the coefficients above vanish identically, except

$$h_4 = \frac{f_4 C_0^2 - 2C_0 C_4 h_0^2 - B_4 h_0^4}{2h_0(C_0^2 + B_0 h_0^2)} \,. \tag{A.6}$$

In particular, all the logarithmic terms vanish and we end up with a usual asymptotic expansion in $1/r$, as expected. Note also that in asymptotically AdS solutions, $C_0 = 0$ as well. Moreover, when there is no superfluid velocity and the isotropy is not broken, $B_4 = 0$ and therefore we end up getting $h_4 = 0$. This last result is useful in making comparisons with the holographic superconductor case investigated in [11].

B On-Shell Action and Counter-Terms

In order to compute the free energy, we need the on-shell action for the type IIB system. As we show below it turns out that, remarkably, the on-shell action can be written purely as a boundary piece, and be easily evaluated. However, this boundary term is divergent: to cancel it we need to introduce boundary counter-terms. In what follows, we describe both these steps.

For the ansatz that we work with, it can be checked directly that, despite the complications of the equations of motion, the following relations hold

$$\mathcal{L}_0 - R = \frac{2L^2}{r^2} T_{yy} = \frac{2L^2}{r^2} T_{zz} \,. \tag{B.1}$$

Here T stands for the stress tensor arising from our IIB Lagrangian, $\mathcal{L}_0$ is defined via

$$S_{IIB} = \int d^5 x \sqrt{-g} \mathcal{L}_0 \,, \tag{B.2}$$

and R is the Ricci scalar. Notice that these relations only depend on our ansatz, i.e. they are true before we use the equations of motion. Going on-shell, we replace T_{yy} and T_{zz} by E_{yy} and E_{zz}, where E denotes the Einstein tensor $E_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R$. Together with the relation

$$E^a{}_a = -\frac{3}{2} R \tag{B.3}$$

that is valid in five dimensions, this implies that

$$\sqrt{-g} \mathcal{L}_0 = \sqrt{-g} \left(\frac{L^2}{r^2} (E_{yy} + E_{zz}) - \frac{2}{3} E^a{}_a \right) \,. \tag{B.4}$$

The right-hand-side depends only on the metric functions and can be evaluated explicitly for our ansatz. Direct computation reveals that it can be written as a total differential so

that the (on-shell) action takes the form

$$S_{IIB,\text{OS}} = -\text{vol}_4 \int_{r_H}^{\infty} dr \left(\frac{2rf(r)}{L^2 h(r)^2} \sqrt{-g} \right)' , \quad \text{where} \quad \sqrt{-g} = \frac{r^3 h(r)}{L^3} \sqrt{\frac{C(r)^2}{f(r)} + B(r)} , \quad \text{(B.5)}$$

and the prime denotes the derivative with respect to r. Because of the presence of f, this expression is zero at the horizon and that end of the integral is safe. But it clearly gets contributions from the boundary, where it diverges as r^4 and we need to regulate it with appropriate counter-terms.

The counter-terms[6] for the gravitational part of the action in asymptotically AdS spaces can be looked up in [29]. Along with these we also have to add counter-terms for the scalar part. The final form of these terms in our notations and conventions can be written as

$$S_{\text{ct}} = 2 \int d^4 x \sqrt{-\gamma} \left(\mathcal{K} - \frac{3}{L} \right) + \int d^4 x \sqrt{-\gamma} \frac{|\psi|^2}{L} . \quad \text{(B.6)}$$

The sign convention for the extrinsic curvature is chosen so that with the outward pointing normal n^a,

$$K_{ab} \equiv \frac{1}{2} (\nabla_a n_b + \nabla_b n_a) . \quad \text{(B.7)}$$

Note that the general gravitational counter-term discussed in [29] involves a boundary Ricci scalar as well: but this does not contribute for us, because our boundary becomes flat as we take it to infinity. The various quantities (including the scalar extrinsic curvature) can be computed by cutting off the spacetime at some finite $r = r_0$, then taking the limit $r_0 \to \infty$ for the quantity $S_{IIB,\text{OS}} + S_{\text{ct}}$ at the end of the computation. If we define the boundary at $r = r_0$, then the outward normal to the surface $\Phi(t, r, x, y, z) \equiv r - r_0 = 0$ is $n_a \sim \nabla_a \Phi$, and after normalizing[7] so that $g^{ab} n_a n_b = 1$, we get

$$n_a = \left(0, \frac{Lh(r)}{r\sqrt{f(r)}}, 0, 0, 0 \right) . \quad \text{(B.8)}$$

Since we need only the scalar extrinsic curvature, we don't need to introduce 4-D coordinates on the boundary and can compute it directly in the bulk coordinates as

$$K = g^{ab} \nabla_a n_b = \frac{f^{1/2} \left(8C^2 + rfB' + 2rCC' + 8Bf + rBf' \right)}{2L(C^2 + Bf)h} . \quad \text{(B.9)}$$

So the final form of the counter-term action is

$$S_{\text{ct}} = \text{Vol}_4 \lim_{r \to \infty} \left[\frac{r^4 f^{1/2} \left(8C^2 + rfB' + 2rCC' + 8Bf + rBf' \right)}{L^5 h \sqrt{C^2 + Bf}} - \frac{r^4}{L^4} \sqrt{C^2 + Bf} \left(\frac{6}{L} - \frac{\psi^2}{L} \right) \right] . \quad \text{(B.10)}$$

---

[6]We loosely refer to the Gibbons-Hawking term also as a counter-term, even though strictly speaking it is a boundary term necessary to make the variational problem well-defined.

[7]Note that the boundary is timelike, so it has a spacelike normal.

With the addition of this piece, the renormalized action $S_{IIB,\text{OS}} + S_{\text{ct}}$ no longer has the r^4 divergence and is finite. The net result is

$$S_{\text{ren}} = \text{vol}_4 \lim_{r \to \infty} \left[\frac{r^4 f^{1/2} \left(8C^2 + rfB' + 2rCC' + 8Bf + rBf' \right)}{L^5 h \sqrt{C^2 + Bf}} + \right.$$

$$\left. - \frac{r^4}{L^5} \sqrt{C^2 + Bf}(6 - \psi^2) - \frac{2r^4 f(r)}{L^5 h(r)} \sqrt{\frac{C(r)^2}{f(r)} + B(r)} \right]. \qquad \text{(B.11)}$$

It is interesting to note that since we are always working with solutions with $\psi_1 = 0$, the scalar piece can in fact be omitted if one desires.

C Superfluid Fraction

In this section we present some details of the definition and computation of the superfluid fraction ζ for our solutions. We start with the renormalized action from the previous appendix and compute the boundary stress tensor and the boundary current by varying with respect to the *boundary* metric and the *boundary* components of the vector potential.

$$\mathcal{T}_{\mu\nu} = \frac{1}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma^{\mu\nu}}, \qquad \mathcal{J}_\mu = \frac{1}{\sqrt{-\gamma}} \frac{\delta S}{\delta A^\mu}, \qquad \text{(C.1)}$$

where now $S = S_{IIB} + S_{\text{ct}}$ with S_{IIB} defined by (2.1) and S_{ct} defined by (B.6). In particular, the relations above are not tied to our ansatz. To compute the boundary stress tensor and current, we need to introduce coordinates on the boundary, and we will use Greek indices for them. After doing the variations, using our ansatz and going on shell on the bulk, the resulting stress tensor and current vanish in the strict $r \to \infty$ limit. This is consistent with the fact that they should be finite since we are using the renormalized action to compute them. The more interesting quantity is the boundary fluid stress tensor and the fluid current, which are defined in AdS_5 via

$$T_{\mu\nu} = \lim_{r \to \infty} r^2 \mathcal{T}_{\mu\nu}, \qquad J_\mu = \lim_{r \to \infty} r^2 \mathcal{J}_\mu. \qquad \text{(C.2)}$$

We are using units where $16\pi G = 1 = L$ in this section. Suppressing the details and restricting to our ansatz these quantities can be explicitly computed in terms of the boundary fall-offs of eqs. (3.18)-(3.21) to be

$$T_{\mu\nu} = \begin{pmatrix} 3f_4 - B_4 & 4C_4 & 0 & 0 \\ 4C_4 & f_4 - 3B_4 & 0 & 0 \\ 0 & 0 & B_4 + f_4 & 0 \\ 0 & 0 & 0 & B_4 + f_4 \end{pmatrix}, \qquad J_\mu = \frac{4}{3} \begin{pmatrix} A_{t,2} \\ A_{x,2} \\ 0 \\ 0 \end{pmatrix}. \qquad \text{(C.3)}$$

Now we follow the interpretation of [21] for these quantities in terms of a two-fluid model on the boundary, where one component is an ordinary (ideal) fluid and the other is a superfluid. First we can write these quantities suggestively in terms of $u_\mu = (-1, 0, 0, 0)$ and $n_\mu = (0, 1, 0, 0)$ as

$$T_{\mu\nu} = (\epsilon + P)\, u_\mu\, u_\nu + P\, \eta_{\mu\nu} - 4B_4\, n_\mu\, n_\nu - 8C_4\, u_{(\mu}\, n_{\nu)}\,, \qquad J_\mu = \rho\, u_\mu - J_s\, n_\mu\,. \qquad \text{(C.4)}$$

where

$$P \equiv f_4 + B_4\,, \qquad \epsilon \equiv 3f_4 - B_4\,, \qquad \rho \equiv -\frac{4}{3}\, A_{t,2}\,, \qquad J_s \equiv -\frac{4}{3}\, A_{x,2}. \qquad \text{(C.5)}$$

Note that what we have done is merely to rewrite the expressions covariantly in terms of the vectors u_μ and n_μ. Another way to state the same thing is that (for example) the most general symmetric second rank tensor constructed from u_μ and n_μ will have to be a linear combination of $\eta_{\mu\nu}$, $u_\mu\, u_\nu$, $u_{(\mu}\, n_{\nu)}$ and $n_\mu\, n_\nu$.

The two fluid model can be defined by the stress tensor

$$T_{\mu\nu} = (\epsilon_0 + P_0)\, u_\mu\, u_\nu + P_0\, \eta_{\mu\nu} + \mu\, \rho_s\, v_\mu\, v_\nu\,, \qquad J_\mu = \rho_n\, u_\mu + \rho_s\, v_\mu\,, \qquad \text{(C.6)}$$

where the subscripts n and s stand for the normal and superfluid components of the charge density, with the total charge density $\rho = \rho_s + \rho_n$. Aside from the various thermodynamical state variables (whose precise interpretations will not be important to us, see [21]), we have also introduced the superfluid velocity v_μ that satisfies the constraint ("Josephson equation")

$$u^\mu\, v_\mu = -1. \qquad \text{(C.7)}$$

The superfluid fraction is defined as

$$\zeta = \frac{\rho_s}{\rho}. \qquad \text{(C.8)}$$

Our stress tensor (C.4) can be brought to the two-fluid form by defining v_μ as

$$v_\mu = u_\mu + \frac{B_4}{C_4}\, n_\mu\,. \qquad \text{(C.9)}$$

This automatically satisfies $v_\mu u^\mu = -1$ as a consequence of $u_\mu u^\mu = -1$, and $n_\mu u^\mu = 0$. Rewriting our stress and current tensors (C.4) in these new variables we get the two-fluid form (C.6):

$$T_{\mu\nu} = (\epsilon + P + 4\, C_4^2 / B_4)\, u_\mu\, u_\nu + P\, \eta_{\mu\nu} - (4\, C_4^2 / B_4)\, v_\mu\, v_\nu\,, \qquad \text{(C.10)}$$

$$J_\mu = (\rho + J_s\, C_4 / B_4)\, u_\mu - (J_s\, C_4 / B_4)\, v_\mu\,. \qquad \text{(C.11)}$$

Reading off the superfluid fraction from this, we find that

$$\zeta = \frac{-(J_s\, C_4 / B_4)}{\rho} = -\frac{A_{x,2}\, C_4}{A_{t,2}\, B_4}\,, \qquad \text{(C.12)}$$

where we have written the final result in terms of the fall-offs obtained directly from the solutions. This is the form we use for making the plots in figure 7.

21

D   The Hairless Solution: Reissner-Nordstrom

In understanding the phase structure, it is important to keep in mind that we are interested in comparing the free energy of the hairy black hole solution to that of Reissner-Nordstrom. In the five dimensional IIB case, the Reissner-Nordstrom metric [11] can be given in terms of our ansatz (3.1) by

$$f(r) = 1 - \frac{1}{r^4}\Big(1 + \frac{4\mu^2}{9}\Big) + \frac{4\mu^2}{9r^6}, \quad A_t = \mu\Big(1 - \frac{1}{r^2}\Big), \tag{D.1}$$

$$h = 1, \quad B = 1, \quad C = 0, \quad A_x = 0, \quad \psi = 0. \tag{D.2}$$

In this notation, the curvature invariants studied in section 5 take the form

$$R = -20f - r\left(10f' + rf''\right), \tag{D.3}$$

$$R_{abcd}\,R^{abcd} = 40f^2 + 4r\,f\left(10f' + r\,f''\right) + r^2\left[22(f')^2 + 8r\,f'\,f'' + r^2(f'')^2\right]. \tag{D.4}$$

All these expressions are obtained after all the necessary rescalings: we have set $16\pi G = L = r_H = 1$. The Hawking temperature now takes the form

$$T_H = \frac{1 - 2\mu^2/9}{\pi}, \tag{D.5}$$

as can be determined by the periodicity of the Euclidean section. The renormalized on-shell action that we determined before takes a simple form for this solution:

$$S_{ren} = 1 + 4\mu^2/9. \tag{D.6}$$

We will compare the free energies of the hairy and hairless cases at the same T/μ to determine which one is the favored phase.

References

[1] S. S. Gubser, "Breaking an Abelian gauge symmetry near a black hole horizon," Phys. Rev. D **78** (2008) 065034 [arXiv:0801.2977 [hep-th]].

[2] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[3] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109].

[4] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[5] S. Weinberg, *The quantum theory of fields, Vol II*, CUP, 1996.

[6] S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, "Building a Holographic Superconductor," Phys. Rev. Lett. **101**, 031601 (2008) [arXiv:0803.3295 [hep-th]].

[7] S. A. Hartnoll, "Lectures on holographic methods for condensed matter physics," Class. Quant. Grav. **26** (2009) 224002 [arXiv:0903.3246 [hep-th]].

[8] C. P. Herzog, "Lectures on Holographic Superfluidity and Superconductivity," J. Phys. A **42** (2009) 343001 [arXiv:0904.1975 [hep-th]].

[9] G. T. Horowitz, "Introduction to Holographic Superconductors", arXiv:1002.1722 [hep-th].

[10] S. S. Gubser and F. D. Rocha, "The gravity dual to a quantum critical point with spontaneous symmetry breaking," Phys. Rev. Lett. **102**, 061601 (2009) [arXiv:0807.1737 [hep-th]].

[11] S. S. Gubser, C. P. Herzog, S. S. Pufu and T. Tesileanu, "Superconductors from Superstrings," Phys. Rev. Lett. **103**, 141601 (2009) [arXiv:0907.3510 [hep-th]].

[12] J. P. Gauntlett, J. Sonner and T. Wiseman, "Holographic superconductivity in M-Theory," Phys. Rev. Lett. **103** (2009) 151601 [arXiv:0907.3796 [hep-th]].

[13] M. Ammon, J. Erdmenger, M. Kaminski and P. Kerner, "Superconductivity from gauge/gravity duality with flavor," Phys. Lett. B **680** (2009) 516 [arXiv:0810.2316 [hep-th]]; "Flavor Superconductivity from Gauge/Gravity Duality," JHEP **0910** (2009) 067 [arXiv:0903.1864 [hep-th]].

[14] S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, "Holographic Superconductors," JHEP **0812**, 015 (2008) [arXiv:0810.1563 [hep-th]].

[15] P. Basu, A. Mukherjee and H. H. Shieh, "Supercurrent: Vector Hair for an AdS Black Hole," Phys. Rev. D **79**, 045010 (2009) [arXiv:0809.4494 [hep-th]].

[16] C. P. Herzog, P. K. Kovtun and D. T. Son, "Holographic model of superfluidity," Phys. Rev. D **79**, 066002 (2009) [arXiv:0809.4870 [hep-th]].

[17] T. Faulkner, G. T. Horowitz, M. M. Roberts, "Holographic quantum criticality from multi-trace deformations," [arXiv:1008.1581 [hep-th]].

[18] V. Keranen, E. Keski-Vakkuri, S. Nowling and K. P. Yogendran, "Inhomogeneous Structures in Holographic Superfluids: II. Vortices," Phys. Rev. D **81** (2010) 126012 [arXiv:0912.4280 [hep-th]].

[19] D. Arean, M. Bertolini, J. Evslin and T. Prochazka, "On Holographic Superconductors with DC Current," JHEP **1007** (2010) 060 [arXiv:1003.5661 [hep-th]].

[20] M. Tinkham, *Introduction to Superconductivity*, 2nd edition, Dover: New York (1996).

[21] J. Sonner and B. Withers, "A gravity derivation of the Tisza-Landau Model in AdS/CFT," arXiv:1004.2707 [hep-th].

[22] D. Arean, P. Basu and C. Krishnan, "The Many Phases of Holographic Superfluids," JHEP **1010**, 006 (2010) [arXiv:1006.5165 [hep-th]].

[23] S. S. Gubser, S. S. Pufu and F. D. Rocha, "Quantum critical superconductors in string theory and M-theory," Phys. Lett. B **683** (2010) 201 [arXiv:0908.0011 [hep-th]].

[24] G. T. Horowitz and M. M. Roberts, "Zero Temperature Limit of Holographic Superconductors," JHEP **0911**, 015 (2009) [arXiv:0908.3677 [hep-th]].

[25] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, "The supergravity dual of N = 1 super Yang-Mills theory," Nucl. Phys. B **569** (2000) 451 [arXiv:hep-th/9909047].

[26] J. Distler and F. Zamora, "Chiral symmetry breaking in the AdS/CFT correspondence," JHEP **0005** (2000) 005 [arXiv:hep-th/9911040].

[27] N. Bobev, N. Halmagyi, K. Pilch and N. P. Warner, "Supergravity Instabilities of Non-Supersymmetric Quantum Critical Points," arXiv:1006.2546 [hep-th].

[28] J. P. Gauntlett, J. Sonner and T. Wiseman, "Quantum Criticality and Holographic Superconductors in M-theory," JHEP **1002** (2010) 060 [arXiv:0912.0512 [hep-th]].

[29] V. Balasubramanian and P. Kraus, "A stress tensor for anti-de Sitter gravity," Commun. Math. Phys. **208**, 413 (1999) [arXiv:hep-th/9902121].